

EVERY EVEN NUMBER OF THE FORM $2n = n^2 + (p + 1)^2 - 2np - d + 1$ CAN BE EXPRESSED AS THE SUM OF TWO PRIMES.

Madieyna Diouf

e-mail: mdiouf1@asu.edu

ABSTRACT. We give a Goldbach's pair for even numbers that are in a specific form. Consider the following two functions where x and m are positive odd integers greater than 1.

- $g(x, m) = x + 2m - (x - m) \bmod(2m)$.
- $f(x)$ returns the smallest prime factor of x .

Let n be a positive odd integer greater than 1. Set p_{max} and d such that

- $p_{max} = \text{maximum of } f(g(n^2 - 2n + 2, p)) \text{ taken over the odd primes } p \leq n$, and
- $d = (n^2 - 2n + 2 - p_{max}) \bmod(2p_{max})$. We observe that every even number of the form $2n = n^2 + (p_{max} + 1)^2 - 2np_{max} - d + 1$ can be expressed as the sum of two primes.

Keywords: Primes, Goldbach's Conjecture.

AMS Classification: Primary 11A41, Secondary 11P32.

1. INTRODUCTION AND STATEMENT OF RESULTS

The historical importance and difficulty of the Even Goldbach Conjecture cannot be overstated. Using Vinogradov's method, Chudakov [2], Van der Corput [3], and Estermann [4] showed that "almost" all even numbers can be written as the sum of two primes. In 1973, Chen Jingrun using the methods of sieve theory showed that every sufficiently large even number can be written as the sum of either two primes, or a prime and a semiprime [6]. Considerable work has been done on Goldbach's weak conjecture that was finally proved in 2013 by Harald Helfgott [5]. His proof directly implies that every even number $n \geq 4$ is the sum of at most four primes [7].

Not succeeding in going as far as the previous authors, we settle for a more modest goal. Using observations that can be easy exercises at a level preceding a first course in elementary number theory, we prove that every even number of the form $2n = n^2 + (p_{max} + 1)^2 - 2np_{max} - d + 1$ can be expressed as the sum of two primes. Moreover, we give the expressions of the two primes in the Goldbach's pair.

Definition. Given a positive odd integer $x > 3$, let A_x denote the set of all odd primes less than or equal to the ceiling of square root of x . That is, $A_x = \{3 \dots \lceil \sqrt{x} \rceil\}$. Define a function

$$(1) \quad g(x, p) = x + 2p - (x - p) \bmod(2p), \text{ where } p \in A_x.$$

For each prime p in A_x , the function $g(x, p)$ yields the smallest odd multiple of p greater than x .

Example 1. For $x = 111$, the set of odd primes less than or equal to $\lceil \sqrt{111} \rceil$ is $A_x = \{3, 5, 7, 11\}$.

Note: The prime 11 is in the set A_x because of the ceiling of square root of x .

$$g(x, p) = x + 2p - (x - p) \bmod(2p) \text{ where } p \text{ is in } A_x.$$

$$g(111, 3) = 111 + 2(3) - (111 - 3) \bmod(2 * 3) = 117.$$

$$g(111, 5) = 111 + 2(5) - (111 - 5) \bmod(2 * 5) = 115.$$

$$g(111, 7) = 111 + 2(7) - (111 - 7) \bmod(2 * 7) = 119.$$

$$g(111, 11) = 111 + 2(11) - (111 - 11) \bmod(2 * 11) = 121.$$

$$g(111, 3) = 117, \text{ is the smallest odd multiple of } 3 \text{ greater than } 111.$$

$$g(111, 5) = 115, \text{ is the smallest odd multiple of } 5 \text{ greater than } 111.$$

$$g(111, 7) = 119, \text{ is the smallest odd multiple of } 7 \text{ greater than } 111.$$

What the function $g(x, p)$ does? The value $(x - p) \bmod(2p)$ is the distance between x and the largest odd multiple of p , less than or equal to x ; therefore, $x - [(x - p) \bmod(2p)]$ is the largest odd multiple of p less than or equal to x . Add a distance $2p$ to the largest odd multiple of p less than or equal to x , this gives the smallest odd multiple of p that is greater than x .

Lemma 1. *Given two positive odd integers x and p , the function $g(x, p)$ yields the smallest odd multiple of p exceeding x .*

Proof. (By contradiction), choose two positive odd integers x and p , and suppose that $g(x, p)$ is not the smallest odd multiple of p that is greater than x . This means that there exists an integer k_2 such that $x < k_2p < g(x, p)$, that is,

$$(2) \quad k_2p < x + 2p - (x - p) \bmod(2p).$$

But since $x - [(x - p) \bmod(2p)]$ is the largest odd multiple of p less than or equal to x , there exists an integer k_1 such that $x - (x - p) \bmod(2p) = k_1p$. Thus, inequality (2) becomes $k_2p < 2p + k_1p$, which implies that $k_2 - k_1 < 2$. This is impossible as $(k_2p \text{ is an odd multiple of } p \text{ that is}) > x$, and $(k_1p \text{ is an odd multiple of } p \text{ that is}) \leq x$, so the value $k_2 - k_1$ cannot be less than 2. Therefore, given two positive odd integers x and p , the function $g(x, p)$ gives rise to the smallest odd multiple of p exceeding x . □

2. EXAMPLE TO ILLUSTRATES THE MAIN RESULT

Example 2. *Let $n = 33$, we have $n^2 - 2n + 2 = 1025$.*

Table 1 on page 3 illustrates how the value of p_{max} is obtained.

- $p_{max} = \max \{f(g(n^2 - 2n + 2, p))\}$ taken over the odd primes $p \leq 33$; thus, $p_{max} = 29$.
- $d = (n^2 - 2n + 2 - p_{max}) \bmod(2p_{max}) = (1025 - 29) \bmod(2 * 29) = 10$.

$$2n = 2 * 33 = 66.$$

$$n^2 + (p_{max} + 1)^2 - 2np_{max} - d + 1 = 33^2 + (29 + 1)^2 - 2 * 33 * 29 - 10 + 1 = 66.$$

Since $2n$ is of the form $n^2 + (p_{max} + 1)^2 - 2np_{max} - d + 1$, we claim that $2n$ can be expressed as a sum of two primes. We prove this argument by showing that

- 1) $k = \frac{g(n^2 - 2n + 2, p_{max})}{p_{max}}$ is a prime. In this example, $k = \frac{1073}{29} = 37$ is a prime. And
- 2) $p_{max} + k = 2n$. That is, $29 + 37 = 66$.

Note. *The positive odd integers 49, 55, 87, 121... are the first examples of n for which $2n$ is not of the form $n^2 + (p_{max} + 1)^2 - 2np_{max} - d + 1$.*

Lemma 2. *Every even number of the form $2n = n^2 + (p_{max} + 1)^2 - 2np_{max} - d + 1$ can be expressed as a sum of a prime p_{max} and an integer $k = \frac{g(n^2 - 2n + 2, p_{max})}{p_{max}}$.*

TABLE 1. $p_{max} = \max \{f(g(n^2 - 2n + 2, p))\}$ taken over the odd primes $p \leq 33$.

$g(n^2 - 2n + 2, p)$	prime factors	$f(g(n^2 - 2n + 2, p))$
$g(1025, 13) = 1027$	13, 79	13
$g(1025, 3) = 1029$	3, 7, 7, 7	3
$g(1025, 7) = 1029$	3, 7, 7, 7	3
$g(1025, 5) = 1035$	3, 3, 5, 23	3
$g(1025, 23) = 1035$	3, 3, 5, 23	3
$g(1025, 17) = 1037$	17, 61	17
$g(1025, 11) = 1045$	5, 11, 19	5
$g(1025, 19) = 1045$	5, 11, 19	5
$g(1025, 29) = 1073$	29, 37	29
$g(1025, 31) = 1085$	5, 7, 31	5

Proof. Suppose that $2n = n^2 + (p_{max} + 1)^2 - 2np_{max} - d + 1$, where n , p_{max} and d are as previously defined on page 1. By the fundamental theorem of arithmetic [1], the prime p_{max} exists. We have

$$\begin{aligned}
2n &= n^2 + (p_{max} + 1)^2 - 2np_{max} - d + 1. \\
2np_{max} &= (p_{max} + 1)^2 + (n - 1)^2 - d. \\
2n &= \frac{(p_{max} + 1)^2 + (n - 1)^2 - d}{p_{max}}. \\
&= \frac{(p_{max} + 1)^2 + (n - 1)^2 - (n^2 - 2n + 2 - p_{max}) \bmod(2p_{max})}{p_{max}}. \\
&= \frac{n^2 - 2n + 2 + 2p_{max} - (n^2 - 2n + 2 - p_{max}) \bmod(2p_{max}) + p_{max}^2}{p_{max}}. \\
&= \frac{g(n^2 - 2n + 2, p_{max}) + p_{max}^2}{p_{max}}. \\
2n &= p_{max} + \frac{g(n^2 - 2n + 2, p_{max})}{p_{max}}.
\end{aligned}$$

□

Lemma 3. *If an even number $2n$ is of the form $n^2 + (p_{max} + 1)^2 - 2np_{max} - d + 1$, then $p_{max} > n - \sqrt{2}\sqrt{n-1}$.*

Proof. Suppose that $2n = n^2 + (p_{max} + 1)^2 - 2np_{max} - d + 1$, where n , p_{max} and d are as previously defined on page 1. In light of Lemma 1, we have $g(n^2 - 2n + 2, p_{max}) = kp_{max}$ is the smallest odd multiple of p_{max} exceeding $n^2 - 2n + 2$. Thus,

$$(3) \quad n^2 - 2n + 2 < kp_{max}.$$

By Lemma 2, we have

$$(4) \quad 2n = p_{max} + k.$$

Relations (3) and (4) give a system

$$(5) \quad \begin{cases} n^2 - 2n + 2 < kp_{max} \\ 2n = p_{max} + k \end{cases}$$

that has a solution $n > 1$ and $n - \sqrt{2}\sqrt{n-1} < p_{max} < n + \sqrt{2}\sqrt{n-1}$ and $2n = p_{max} + k$, using *Walfram Mathematica*.

Thus, if $2n$ is of the form $n^2 + (p_{max} + 1)^2 - 2np_{max} - d + 1$, then $p_{max} > n - \sqrt{2}\sqrt{n-1}$. \square

Lemma 4. *If $p_{max} > n - \sqrt{2}\sqrt{n-1}$, then $k = \frac{g(n^2-2n+2, p_{max})}{p_{max}}$ is a prime.*

Proof. Suppose that $p_{max} > n - \sqrt{2}\sqrt{n-1}$. In virtue of Lemma 1, $g(n^2-2n+2, p_{max}) = kp_{max}$, where k is an odd integer. The objective is to show that $k = \frac{g(n^2-2n+2, p_{max})}{p_{max}}$ is a prime.

Case 1: If n is a prime,

then n is the smallest prime factor of n^2 . Thus, $n = \max\{f(g(n^2 - 2n + 2, p))\}$ taken over the odd primes $p \leq n$. That is, $p_{max} = n$. We obtain

$$1) \ g(n^2 - 2n + 2, p_{max}) = kp_{max}.$$

$$2) \ g(n^2 - 2n + 2, n) = n^2 - 2n + 2 + 2n - \text{mod}(n^2 - 2n + 2 - n, 2n).$$

$$\begin{aligned} g(n^2 - 2n + 2, n) &= n^2 + 2 - \text{mod}(n^2 - 3n + 2, 2n). \\ &= n^2. \end{aligned}$$

Since $p_{max} = n$, we have $g(n^2 - 2n + 2, p_{max}) = g(n^2 - 2n + 2, n)$. Observations 1) and 2) imply that

$$\begin{aligned} kp_{max} &= n^2. \\ kn &= n^2. \\ k &= n = p_{max} \text{ is a prime.} \end{aligned}$$

Case 2: Say n is not a prime.

1) We have $g(n^2 - 2n + 2, p_{max}) = kp_{max}$ where p_{max} is the smallest prime factor of $g(n^2 - 2n + 2, p_{max})$. Hence, k does not have a prime factor less than p_{max} .

2) We also have,

$$\begin{aligned} p_{max} &> n - \sqrt{2}\sqrt{n-1}, \text{ in our hypothesis.} \\ \text{Thus, } p_{max}^3 &> (n - \sqrt{2}\sqrt{n-1})^3. \text{ We know that,} \\ (n - \sqrt{2}\sqrt{n-1})^3 &> n^2 \text{ when } n > 7. \text{ Hence,} \end{aligned}$$

$$(6) \quad p_{max}^3 > n^2 \text{ when } n > 7.$$

Furthermore, by definition,

$$g(n^2 - 2n + 2, p_{max}) = n^2 - 2n + 2 + 2p_{max} - \text{mod}(n^2 - 2n + 2 - p_{max}, 2p_{max}).$$

$$g(n^2 - 2n + 2, p_{max}) \leq n^2 \text{ as } p_{max} \leq n. \text{ But since } n \text{ is not a prime,}$$

$$(7) \quad g(n^2 - 2n + 2, p_{max}) < n^2 \text{ as } p_{max} < n. \quad .$$

$$\begin{aligned}
(6) \text{ and } (7) \text{ imply that, } p_{max}^3 &> n^2 > g(n^2 - 2n + 2, p_{max}) \text{ when } n > 7. \\
p_{max}^3 &> kp_{max} \text{ because } g(n^2 - 2n + 2, p_{max}) = kp_{max}. \\
p_{max}^2 &> k. \\
p_{max} &> \sqrt{k} \text{ when } n > 7.
\end{aligned}$$

Part 1) implies that k does not have a prime factor less than p_{max} .

Part 2) implies that $p_{max} > \sqrt{k}$ when $n > 7$.

Part 1) and 2) imply that k does not have any prime factor less than \sqrt{k} . Hence, k is a prime. \square

Theorem 1. *Every even number of the form $2n = n^2 + (p_{max} + 1)^2 - 2np_{max} - d + 1$ can be expressed as the sum of two primes.*

Proof. In Lemma 2 we show that $2n = p_{max} + \frac{g(n^2 - 2n + 2, p_{max})}{p_{max}}$. Lemma 3 and 4 imply that $\frac{g(n^2 - 2n + 2, p_{max})}{p_{max}}$ is a prime. Therefore, $2n$ can be expressed as the sum of two primes. \square

REFERENCES

- [1] Gauss, Carl Friedrich; Clarke, Arthur A. (translator into English) (1986), *Disquisitiones Arithmeticae (Second, corrected edition)*, New York: Springer, ISBN 978-0-387-96254-2
- [2] Chudakov, Nikolai G. (1937). *[On the Goldbach problem]*. Doklady Akademii Nauk SSSR. 17: 335338.
- [3] Van der Corput, J. G. (1938). *Sur l'hypothèse de Goldbach* (PDF). Proc. Akad. Wet. Amsterdam (in French). 41: 7680.
- [4] Estermann, T. (1938). *On Goldbach's problem: proof that almost all even positive integers are sums of two primes*. Proc. London Math. Soc. 2. 44: 307314.
- [5] Helfgott, H. A. (2013). *The ternary Goldbach conjecture is true*. arXiv:1312.7748
- [6] Chen, J. R. (1973). *On the representation of a larger even integer as the sum of a prime and the product of at most two primes*. Sci. Sinica. 16: 157176.
- [7] Wikipedia *Goldbach's Conjecture* (2017) https://en.wikipedia.org/wiki/Goldbach's_conjecture#Rigorous_results [Online; accessed 08-March-2017]